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PROPAGATION IN WAVE GUIDES BOUNDED BY ELECTRICALLY ANISOTROPIC PLATES

by

BERNARD FRIEDMAN AND HENRY MALIN

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by

Bernard Friedman and Henry Malin

Morris Kline
Project Director

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ABSTRACT

In this paper, we study the propagation of waves in non-isotropic plates. If along two principal directions, the plate has a common dielectric constant, Maxwell's equations can be reduced to a wave equation for one of the components of the field. It is found that if the plates are used as the boundaries of a wave guide, modes similar to those given by Brillouin: "Spiraled Coils as Wave Guides", Cruft Laboratory, Harvard University, are obtained. This procedure enables us to take into account the finite conductivity and thickness of the plate. It is found that these effects are of second order with respect to the frequency.

1. Introduction.

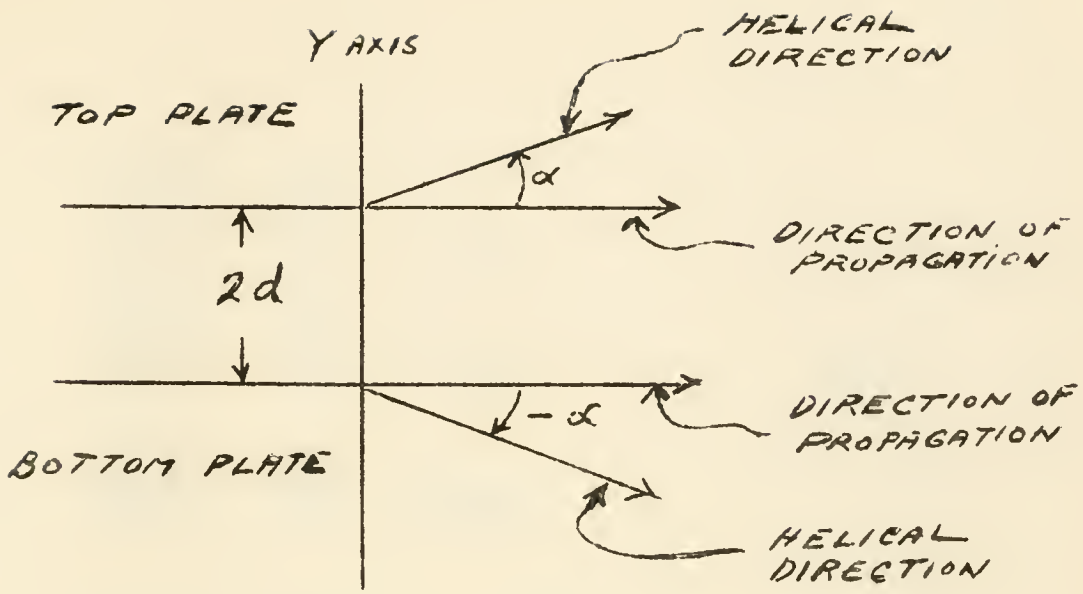
The object of this note is to study an idealized helical guide whose associated boundary conditions seem to us more natural than those which have been used heretofore. The idea which we exploit of using a thick walled guide of non-isotropic material can be adapted to various geometrical shapes. It is most convenient to make use of the simple geometric configuration due to Brillouin: "Spiraled Coils as Wave Guides", Cruft Laboratory, Harvard University. The helix is replaced by a sheet or plate of non-isotropic material on which parallel lines are assumed to give a helical direction. In this material the conductivity is assumed to be large in the helical direction and zero in the two other perpendicular directions. We have assumed continuity of the field components and have investigated the modes of this structure. The analogy is here with crystal optics where the index of refraction varies in different directions. By taking axes along the principal directions and assuming that two of the dielectric constants are equal we are able to obtain the wave equation for the component of the field in the helical direction. The field components in the other directions may then be derived in terms of these.

In the present work two cases are discussed. In the first case, it will be assumed that the plates have infinite thickness but are not perfect conductors. In the second case the plates are of finite thickness but of infinite conductivity. The two cases differ only in the initial conditions that are applied to the two problems.

While it has seemed very worth while to us and to other workers, to scrutinize carefully the type of boundary conditions usually associated with helical wave guides, it is perhaps somewhat gratifying that modifications in boundary conditions appear to give rise to second order effects only.

2. Derivation of the Field inside the Plates.

We shall assume that the two plates are bounded by the two planes $y = \pm d$. It is convenient to choose axes along the principal directions of the plates. For this reason we introduce two sets of axes, so that the z-axis of each



set is the helical direction on the corresponding plate. The y-axis is the same in each set, the x-axis is the appropriate perpendicular.

The electromagnetic components must be found in two different media; (i.e.) in the region between the plates which is assumed to be air and in the region occupied by the plates. The latter components will be derived and then the components in air will be deduced from these by the specialization of the dielectric constants. We shall show that starting with Maxwell's equations, the components of the field in the plate satisfy a wave equation. Hence these components are known, and the others can be deduced from them.

The pertinent Maxwell equations, with time dependence harmonic, which will be used are:

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -i\omega\mu H_x$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -i\omega\mu H_y \qquad (2.1)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -i\omega\mu H_z \quad ,$$

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = i\omega\epsilon E_x$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = i\omega\epsilon E_y \qquad (2.2)$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = i\omega\epsilon_3 E_z$$

and

$$\epsilon \frac{\partial E_x}{\partial x} + \epsilon \frac{\partial E_y}{\partial y} + \epsilon_3 \frac{\partial E_z}{\partial z} = 0 . \qquad (2.3)$$

We call attention to the presence of ϵ_3 in Eqs. (2.2) and (2.3). This is an explicit formulation of our basic assumption that the dielectric constant in the z-direction is different from the dielectric constants in the x and y directions which are assumed to be equal.

The electric field component will now be found. Multiply the last of Eqs. (2.2) by $-i\omega\mu$; differentiate the second of Eqs. (2.1) with respect to x, the first with respect to y and subtract. We get

$$\omega^2 \epsilon_3 \mu E_z = -i\omega\mu\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \qquad (2.4)$$

$$= \frac{\partial^2 E_x}{\partial z \partial x} - \frac{\partial^2 E_z}{\partial x^2} - \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_y}{\partial z \partial y}$$

Eq. (2.4) may be re-written with the aid of Eq. (2.3) as

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\epsilon_3}{\epsilon} \frac{\partial^2 E_z}{\partial z^2} + \omega^2 \epsilon_3 \mu E_z = 0. \qquad (2.5)$$

It is now assumed that the field components of E and H in the upper plate travel in planes parallel to the y-axis. Thus we assume for E_z an expression of the form

$$E_z = \bar{f}_3(y)\ e^{i\beta(z \sin\alpha + x \cos\alpha)} \qquad (2.6)$$

where $\bar{f}_3(y)$ is a function of y to be determined. Substitution of Eq. (2.6) into Eq. (2.5) yields for $\bar{f}_3$ the following second order linear differential equation:

$$\frac{d^2\bar{f}_3}{dy^2} + k_1^2 \bar{f}_3 = 0. \tag{2.7}$$

where

$$k_1^2 = \omega^2 \epsilon_3 \mu - \beta^2 + \beta^2(1 - \frac{\epsilon_3}{\epsilon}) \sin^2\alpha \quad . \tag{2.8}$$

Hence

$$\bar{f}_3(y) = A \cos k_1 y + B \sin k_1 y$$

$$E_z = (A \cos k_1 y + B \sin k_1 y) e^{i\beta (z \sin\alpha + x \cos\alpha)} \tag{2.9}$$

One easily derives the following equation

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial z^2} - \frac{\partial}{\partial y}\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} \right) + \omega^2 \epsilon \mu E_y = 0 ; \tag{2.10}$$

Employing Eq. (2.3), Eq. (2.10) reduces to

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} - (1 - \frac{\epsilon_3}{\epsilon}) \frac{\partial^2 E_z}{\partial y \partial z} + \omega^2 \epsilon \mu E_y = 0. \tag{2.11}$$

Hence if we assume

$$E_y = \bar{f}_2(y) e^{i\beta (z \sin\alpha + x \cos\alpha)} \tag{2.12}$$

the function $\bar{f}_2(y)$ satisfies the differential equation

$$\frac{d^2\bar{f}_2}{dy^2} + k^2 \bar{f}_2 = i\beta (1 - \frac{\epsilon_3}{\epsilon}) \frac{d\bar{f}_3}{dy} \sin\alpha \tag{2.13}$$

where

$$k^2 = \omega^2 \epsilon \mu - \beta^2 \quad . \tag{2.14}$$

Eq. (2.13) has for its solution

$$\bar{f}_2(y) = C \cos ky + D \sin ky + i \beta k_1 \sin\alpha \frac{(1 - \frac{\epsilon_3}{\epsilon})}{k_1^2 - k^2} (A \sin k_1 y - B \cos k_1 y) \tag{2.15}$$

Finally, the same method shows that if

$$E_x = \bar{f}_1(y)\, e^{i \beta (z \sin\alpha + x \cos\alpha)} \quad , \tag{2.16}$$

the function $f_1(y)$ satisfies the differential equation

$$\frac{d^2 \bar{f}_1}{dy^2} + k^2 \bar{f}_1 = - (1 - \frac{\epsilon_3}{\epsilon}) \beta^2 \sin\alpha \cos\alpha \, \bar{f}_3(y) \tag{2.17}$$

Thus

$$\bar{f}_1(y) = G \cos ky + F \sin ky + \frac{(1 - \frac{\epsilon_3}{\epsilon})}{k_1^2 - k^2} \beta^2 \sin\alpha \cos\alpha (A \cos k_1 y + B \sin k_1 y). \tag{2.18}$$

The constants G, F, C and D are related. The use of Eq. (2.3) shows that

$$G = - \frac{D k}{i \beta \cos\alpha} \quad , \qquad F = \frac{C k}{i \beta \cos\alpha} \quad , \tag{2.19}$$

which when used in Eq. (2.18) yields

$$\bar{f}_1(y) = - \frac{D k}{i \beta \cos\alpha} \cos ky + \frac{C k}{i \beta \cos\alpha} \sin ky + \frac{(1 - \frac{\epsilon_3}{\epsilon})}{k_1^2 - k^2} \beta^2 \sin\alpha \cos\alpha (A \cos k_1 y + B \sin k_1 y) \tag{2.20}$$

The components of the magnetic field are obtained from Eqs.(2.1). We find that if $\bar{g}_1$, $\bar{g}_2$, $\bar{g}_3$ bear the same relationship to the magnetic components that $\bar{f}_1$, $\bar{f}_2$ and $\bar{f}_3$ do to the electric components, then

$$-i\omega\mu\bar{g}_1 = -i\beta\sin\alpha(C\cos ky + D\sin ky)$$

$$+ k_1\omega^2\epsilon\mu\frac{(1-\frac{\epsilon_3}{\epsilon})}{k_1^2-k^2}(A\sin k_1y - B\cos k_1y) \qquad (2.21)$$

$$-i\omega\mu\bar{g}_2 = k\tan\alpha(C\sin ky - D\cos ky)$$

$$+ i\beta\omega^2\epsilon\mu\cos\alpha\frac{(1-\frac{\epsilon_3}{\epsilon})}{k_1^2-k^2}(A\cos k_1y + B\sin k_1y) \qquad (2.22)$$

$$-i\omega\mu\bar{g}_3 = -\frac{\beta^2\cos^2\alpha+k^2}{i\beta\cos\alpha}(C\cos ky + D\sin ky)\,. \qquad (2.23)$$

Eqs. (2.9),(2.15),(2.20),(2.21),(2.22) and (2.23) represent the E and H fields in the top plate.

The equations for the field in the bottom plate may be derived in the same manner as those in the top plate. However, it may be observed that they may be obtained from those of the top plate by replacing α by $-\alpha$.

We have for the E and H fields in the bottom plate the following six equations

$$\underline{f_1} = -\frac{D_1k}{i\beta\cos\alpha}\cos ky - \frac{C_1k\sin ky}{i\beta\cos\alpha}$$

$$-\beta^2\sin\alpha\cos\alpha\frac{(1-\frac{\epsilon_3}{\epsilon})}{k_1^2-k^2}(A_1\cos k_1y - B_1\sin k_1y) \qquad (2.24)$$

$$\underline{f}_2 = C_1 \cos ky - D_1 \sin ky + k_1 i\beta \sin\alpha \frac{(1-\frac{\epsilon_3}{\epsilon})}{k_1^2 - k^2} (A_1 \sin k_1 y + B_1 \cos k_1 y) \qquad (2.25)$$

$$\underline{f}_3 = A_1 \cos k_1 y - B_1 \sin k_1 y \, . \qquad (2.26)$$

$$-i\omega\mu \underline{g}_1 = i\beta \sin\alpha (C_1 \cos ky - D_1 \sin k_1 y)$$

$$-k_1 \omega^2 \epsilon \mu \frac{(1-\frac{\epsilon_3}{\epsilon})}{k_1^2 - k^2} (A_1 \sin k_1 y + B_1 \cos k_1 y) \qquad (2.27)$$

$$-i\omega\mu \underline{g}_2 = k \tan\alpha (C_1 \sin ky + D_1 \cos ky)$$

$$+ i\beta \omega^2 \epsilon \mu \cos\alpha \frac{(1-\frac{\epsilon_3}{\epsilon})}{k_1^2 - k^2} (A_1 \cos k_1 y - B_1 \sin k_1 y) \qquad (2.28)$$

$$-i\omega\mu \underline{g}_3 = -\frac{\beta^2 \cos^2\alpha + k^2}{i\beta \cos\alpha} (C_1 \cos ky - D_1 \sin k_1 y) \, . \qquad (2.29)$$

3. Derivation of the Field Components in the Region between the Two Plates.

In the region between the two plates (air) the dielectric constants are equal. The appropriate Maxwell equations are Eqs.(2.1),(2.2) and (2.3) with ϵ_3 replaced by ϵ. The field components are thus derivable from those of the preceding section on replacing ϵ_3 by ϵ. It is not to be supposed, however, that the quantity $\frac{1 - \frac{\epsilon_3}{\epsilon}}{k_1^2 - k^2}$ is zero since in this case, $k = k_1$. Indeed the use of Eqs. (2.08) and (2.14) shows that

$$\frac{1-\frac{\epsilon_3}{\epsilon}}{k_1^2 - k^2} = \frac{1}{\beta^2 \sin^2\alpha - \omega^2 \epsilon \mu} \qquad (3.1)$$

If now, in the relevant equations of paragraph 2, ϵ_3 be taken equal to ϵ and k equal to k_1, the field components of E and H may, after some calculations, be exhibited in the following convenient form

$$f_1 = C_2 \cos ky + D_2 \sin ky \tag{3.2}$$

$$f_2 = \frac{i\beta}{k}(D_2 \cos\alpha + B_2 \sin\alpha)\cos ky - \frac{i\beta}{k}(C_2\cos\alpha + A_2\sin\alpha)\sin ky \tag{3.3}$$

$$f_3 = A_2 \cos ky + B_2 \sin ky. \tag{3.4}$$

$$-i\omega\mu g_1 = \frac{\cos ky}{k}\left(D_2\beta^2\sin\alpha\cos\alpha + B_2(k^2+\beta^2\sin^2\alpha)\right) - \frac{\sin ky}{k}\left(C_2\beta^2\sin\alpha\cos\alpha + A_2(\beta^2\sin\alpha\cos\alpha + k^2)\right) \tag{3.5}$$

$$-i\omega\mu g_2 = i\beta\cos ky\,(C_2\sin\alpha - A_2\cos\alpha) + i\beta\sin ky\,(D_2\sin\alpha - B_2\cos\alpha) \tag{3.6}$$

$$-i\omega\mu g_3 = -\frac{\cos ky}{k}\left(D_2(k^2+\beta^2\cos^2\alpha) + B_2\beta^2\sin\alpha\cos\alpha\right) + \frac{\sin ky}{k}\left(C_2(k^2+\beta^2\cos^2\alpha) + A_2\beta^2\sin\alpha\cos\alpha\right) \tag{3.7}$$

The above formulas give the field components in the direction of the axes on the top plate. Later we shall need the field components in the direction of the axes on the bottom plate. They will be obtained from formulas (3.2) - (3.7) simply by replacing α by $-\alpha$.

In our subsequent work we shall need E_y and H_y in the form

$$\text{constant} \cdot \sin ky. \tag{3.8}$$

This condition requires that

$$D_2 \cos\alpha + B_2 \sin\alpha = 0$$

$$C_2 \sin\alpha - A_2 \cos\alpha = 0$$

The Eqs. (3.2) - (3.7) may now be written as:

$$f_1 = A_3 \cos\alpha \cos ky - B_3 \sin\alpha \sin ky \tag{3.10}$$

$$f_2 = -\frac{i\beta}{k} A_3 \sin ky \tag{3.11}$$

$$f_3 = A_3 \sin\alpha \cos ky + B_3 \cos\alpha \sin ky \tag{3.12}$$

$$-i\omega\mu g_1 = -A_3 \frac{\omega^2 \epsilon \mu \sin\alpha}{k} \sin ky + B_3 k \cos\alpha \cos ky \tag{3.13}$$

$$-i\omega\mu g_2 = -B_3 i\beta \sin ky \tag{3.14}$$

$$-i\omega\mu g_3 = A_3 \frac{\omega^2 \epsilon \mu}{k} \cos\alpha \sin ky + B_3 k \sin\alpha \cos ky \,. \tag{3.15}$$

A result due to Brillouin (loc.cit) will now be deduced from Eqs. (3.10)-(3.15). Brillouin assumes two ideal sheets of wires at angles $\pm\alpha$ to the z-axis situated in the planes $y = \pm d$. The boundary conditions are the vanishing of the electric and magnetic components in the helical (that is to say z) direction. The boundary conditions require

$$f_3 = 0, \; g_3 = 0 \quad \text{at } y = \pm d. \tag{3.16}$$

Using the above equations at y = d we find

$$A_3 \sin\alpha \cos kd + B_3 \cos\alpha \sin kd = 0$$

$$A_3 \frac{\omega^2 \epsilon \mu}{k} \cos\alpha \sin kd + B_3 k \sin\alpha \cos kd = 0. \tag{3.17}$$

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It is to be observed that the replacement of d by -d, α by $-\alpha$ leaves Eqs. (3.17) unchanged. Thus Eqs. (3.17) automatically takes care of the boundary condition at y = -d. Eq. (3.17) gives rise to

$$\frac{\tan^2\alpha}{\omega^2\epsilon\mu} = \frac{\tan^2 kd}{k^2} \tag{3.18}$$

which is the desired transcendental equation for the propagation constants obtained by Brillouin.

4. Plates of Infinite Thickness.

The first application of the results of paragraphs 2 and 3 will be to plates of infinite thickness and finite conductivity. The field components in the upper plate must behave like outgoing waves at infinity. This will be accomplished if the constants in Eq. (2.9) and (2.20) are required to satisfy:

$$B = iA, \qquad D = iC \tag{4.1}$$

Making this change and using Eq. (3.1) we find that the E and H vectors in the top plate are given by:

$$\begin{aligned}
\bar{f}_1 &= A\,\frac{\beta^2\sin\alpha\cos\alpha}{\beta^2\sin^2\alpha - \omega^2\epsilon\mu}\,e^{ik_1 y} - C\,\frac{k}{\beta\cos\alpha}\,e^{iky} \\
\bar{f}_2 &= A\,\frac{k_1\beta\sin\alpha}{\beta^2\sin^2\alpha - \omega^2\epsilon\mu}\,e^{ik_1 y} + C\,e^{iky} \\
\bar{f}_3 &= A\,e^{ik_1 y} \\
-i\mu\omega\bar{g}_1 &= -A\,\frac{k_1 i\omega^2\epsilon\mu}{\beta^2\sin^2\alpha - \omega^2\epsilon\mu}\,e^{ik_1 y} - C\,i\beta\sin\alpha\,e^{iky} \\
-i\mu\omega\bar{g}_2 &= A\,\frac{i\beta\cos\alpha\,\omega^2\epsilon\mu}{\beta^2\sin^2\alpha - \omega^2\epsilon\mu}\,e^{ik_1 y} - C\,k\,i\tan\alpha\,e^{iky} \\
-i\mu\omega g_3 &= -C\,\frac{\beta^2\cos^2\alpha + k^2}{i\beta\cos\alpha}\,e^{iky}\,.
\end{aligned} \tag{4.2}$$

Now let $B_1 = iA_1$, $D_1 = iC_1$, the components for the bottom plate are given by:

$$\underline{f}_1 = -A_1 \frac{\beta^2 \sin\alpha\cos\alpha}{(\beta^2\sin^2\alpha - \omega^2\epsilon\mu)} e^{-ik_1 y} - C_1 \frac{k}{\beta\cos\alpha} e^{-iky}$$

$$\underline{f}_2 = -A_1 \frac{k\beta\sin\alpha}{(\beta^2\sin^2\alpha - \omega^2\epsilon\mu)} e^{-ik_1 y} + C_1 e^{-iky}$$

$$\underline{f}_3 = A_1 e^{-ik_1 y}$$

(4.3)

$$-i\mu\omega\underline{g}_1 = -A_1 \frac{k_1\omega^2\epsilon\mu\, i}{\beta^2\sin^2\alpha - \omega^2\epsilon\mu} e^{-ik_1 y} + C_1 \beta \sin\alpha\, e^{-iky}$$

$$-i\mu\omega\underline{g}_2 = A_1 \frac{i\beta\omega^2\epsilon\mu\cos\alpha}{\beta^2\sin^2\alpha - \omega^2\epsilon\mu} e^{-ik_1 y} + C_1 k\, i \tan\alpha\, e^{-iky}$$

$$-i\mu\omega\underline{g}_3 = -C_1 \frac{\beta^2\cos^2\alpha + k^2}{i\beta\cos\alpha} e^{-iky}$$

It is important for the sequel that one can go from Eq. (4.2) to Eq. (4.3) by the transformation $k \to -k$, $k_1 \to -k_1$, $\alpha \to -\alpha$.

The fields are matched at the boundaries by the following conditions:

Continuity of the tangential components of E and H in the helical direction. (4.4)

Continuity of the tangential components of E and H in the direction perpendicular to the helical direction. (4.5)

The boundary conditions require that the tangential components of E and H be continuous. Conditions (4.4) and (4.5) may therefore be replaced by

Continuity of the tangential components of E and H in the direction of propagation.

Continuity of the tangential components of E and H in the direction perpendicular to the direction of propagation.

In order to simplify calculations, we will use the boundary conditions in the form

$$H_{\perp} - \frac{\omega^2\epsilon\mu}{k} \tan kd\, E_{\parallel}\Big|_{\text{air}} = H_{\perp} - \frac{\omega^2\epsilon\mu}{k} \tan kd\, E_{\parallel}\Big|_{\text{plate}}, \quad (4.6)$$

$$H_{\parallel} - k \cot kd\, E_{\perp}\Big|_{\text{air}} = H_{\parallel} - k \cot kd\, E_{\perp}\Big|_{\text{plate}}, \quad (4.7)$$

where $E_{\parallel}$, $H_{\parallel}$ ($E_{\perp}$, $H_{\perp}$) are the components of E and H parallel (perpendicular) to the direction of propagation. The boundary conditions (4.6) and (4.7) are to hold at $y = \pm d$.

In the air, the desired components are as follows:

$$E_{\parallel} : f_1 \cos\alpha + f_3 \sin\alpha = A_3 \frac{ik}{\beta} \cos ky, \quad (4.8)$$

$$E_{\perp} : -f_1 \sin\alpha + f_3 \cos\alpha = B_3 \frac{i}{\beta} \sin ky, \quad (4.9)$$

$$H_{\parallel} : -i\mu\omega(g_1 \cos\alpha + g_3 \sin\alpha) = B_3 \frac{ik}{\beta} \cos ky, \quad (4.10)$$

$$H_{\perp} : -i\mu\omega(-g_1 \sin\alpha + g_3 \cos\alpha) = A_3 \frac{i\omega^2\epsilon\mu}{\beta} \sin ky; \quad (4.11)$$

and in the top plate:

$$E_{\parallel} : \bar{f}_1 \cos\alpha + \bar{f}_3 \sin\alpha = -A \frac{k^2 \sin\alpha}{\beta^2\sin^2\alpha - \omega^2\epsilon\mu} e^{ik_1 y} - E \frac{k}{\beta} e^{iky}, \quad (4.12)$$

$$E_{\perp} : -\bar{f}_1 \sin\alpha + \bar{f}_3 \cos\alpha = -A \frac{\omega^2\epsilon\mu \cos\alpha}{\beta^2\sin^2\alpha - \omega^2\epsilon\mu} e^{ik_1 y} + E \frac{k \tan\alpha}{\beta} e^{iky} \quad (4.13)$$

$$H_{\parallel} : -i\mu\omega(\bar{g}_1 \cos\alpha + \bar{g}_3 \sin\alpha) = -A \frac{ik_1\omega^2\epsilon\mu\cos\alpha}{\beta^2\sin^2\alpha - \omega^2\epsilon\mu} e^{ik_1 y} + E \frac{ik^2\tan\alpha}{\beta} e^{iky} \quad (4.14)$$

$$H_{\perp} : -i\mu\omega(-\bar{g}_1 \sin\alpha + \bar{g}_3 \cos\alpha) = A \frac{ik_1\omega^2\epsilon\mu\sin\alpha}{\beta^2\sin^2\alpha - \omega^2\epsilon\mu} e^{ik_1 y} + E \frac{i\omega^2\epsilon\mu}{\beta} e^{iky}. \quad (4.15$$

The equations for the bottom plate corresponding to Eqs. (4.12)-(4.15) are gotten from them by replacing α, k_1 and k by their negatives.

Using the boundary conditions, and denoting by k^*, for the moment, the free k's, which occur in the Eqs. (4.12)-(4.15) for the top plate we find that

$$H_\perp - \frac{\omega^2\epsilon\mu}{k} \tan kd \; E_{||} = 0 = A\omega^2\epsilon\mu\sin\alpha \frac{(ik_1 + \frac{k^{*2}}{k} \tan kd)}{\beta^2\sin^2\alpha - \omega^2\epsilon\mu} e^{ik_1 d}$$

$$+ C \frac{\omega^2\epsilon\mu}{\beta} (i + \frac{k^*}{k} \tan kd) e^{ikd} \qquad (4.16)$$

$$H_{||} - k \cot kd \; E_\perp = 0 = -A\omega^2\epsilon\mu\cos\alpha \frac{(ik_1 - k \cot kd)}{\beta^2\sin^2\alpha - \omega^2\epsilon\mu} e^{ik_1 d}$$

$$+ C \frac{\tan\alpha}{\beta} (k^2 i - kk^* \cot kd) e^{ikd} \qquad (4.17)$$

It must now be shown that Eqs. (4.16) and (4.17) are the same as those which would have been obtained if we had set up the boundary conditions on the bottom plate. Let us replace the d in the functions tan kd, cot kd by -d. This yields the field components on the bottom plate. Furthermore, let us replace α, k_1 and k^* by their negatives. This, as has been remarked, yields the components in the direction of the axis on the bottom plate. If this be done and A_1 and C_1 of Eqs. (4.3) be taken as

$$A_1 = A e^{2ik_1 d}, \quad C_1 = C e^{2ikd}, \qquad (4.18)$$

the Eqs. (4.16) and (4.17) remain unchanged.

In order that Eqs. (4.16) and (4.17) have non-zero solutions in A and C, it is necessary and sufficient that the determinant formed by their

coefficients be zero. Thus, letting $\delta = \frac{k^*}{k}$, we get

$$\begin{vmatrix} \omega^2\epsilon\mu \sin\alpha \dfrac{(ik_1 + \frac{k^{*2}}{k} \tan kd)}{\beta^2\sin^2\alpha - \omega^2\epsilon\mu} e^{ik_1 d} & \dfrac{\omega^2\epsilon\mu}{\beta} (i + \delta \tan kd)\, e^{ikd} \\ -\omega^2\epsilon\mu\cos\alpha \dfrac{(ik_1 - k \cot kd)}{\beta^2\sin^2\alpha - \omega^2\epsilon\mu} e^{ik_1 d} & \dfrac{k^2\tan\alpha}{\beta} (i - \delta \cot kd)\, e^{ikd} \end{vmatrix} = 0 \quad (4.19)$$

Eq. (4.19) has the factors $\omega^2\epsilon\mu$, $e^{ik_1 d}$, e^{ikd}, β^{-1}, $\beta^2\sin^2\alpha - \omega^2\epsilon\mu$ and $1 + \delta \tan kd$. Suppressing these factors we are led to the following equation for the determination of k:

$$\frac{ik_1}{k}\left[i\,\delta\, k^2 \tan^2\alpha + \omega^2\epsilon\mu \tan kd \right] + i\,\delta\, k^{*2}\tan^2\alpha \tan kd - \omega^2\epsilon\mu = 0 \quad (4.20)$$

Let us observe again that the replacement of k_1 by $-k_1$, α by $-\alpha$, d by -d and δ by $-\delta$ leaves Eqs. (4.20) unchanged. Now replace $\delta = \frac{k^*}{k}$ by unity, k by iK and Kd by x. We get

$$\frac{k_1 d}{x}\left[-x^2 \tan^2\alpha + \omega^2\epsilon\mu d^2 \tanh x \right] = x^2 \tan^2\alpha \tanh x - \omega^2\epsilon\mu d^2 \quad (4.21)$$

In the case of infinite conductivity (k_1 is infinite), Eq. (4.21) reduces to

$$-x^2 \tan^2\alpha + \omega^2\epsilon\mu d^2 \tanh x = 0. \quad (4.22)$$

This is precisely the equation that has already been obtained for this case under different boundary conditions by W. W. Harrison: 'Effect of Choice of Boundary

Conditions on Propagation of Power in the Traveling Wave Tube"- Contract No. Nobsr 39160 - Sylvania Electric Products, Inc. Research Department, Flushing, N. Y.

In the case of finite (but large) conductivity the right hand side of Eq. (4.21) serves as a corrective term in the presence of conductivity. Eq. (4.21), for this case, can be solved by the method of successive approximations.

A numerical example has been worked out to determine the attenuation of the guide with apparent disappointing results. The details will not be given here except that in the range of parameters of interest, the left hand side of Eq. (4.21) when set equal to zero has a root x for which the tanh x is very nearly one. It is then seen that this value of x makes the right hand side very small independent of the value of k_1 (the conductivity). Physically, it seems to be the case that this method gives results which are second order effects compared with the frequency.

5. Plates of Finite Thickness.

We now investigate plates which are perfect conductors in the helical direction but have a finite thickness t.

The boundary conditions which will be taken for this case are:

Vanishing of the tangential components of E and H in the helical direction at $y = \pm (d + t)$, (5.1)

Continuity of the tangential components of E and H in the helical direction at $y = \pm d$, and (5.2)

Continuity of the tangential component of H in the direction perpendicular to the helical direction at $y = \pm d$. (5.3)

For a conductor in which $k_1 \to \infty$, Eq. (2.21) implies that

$$A \sin k_1 y \to B \cos k_1 y \text{ or } B \to A \tan k_1 y. \tag{5.4}$$

Using Eq. (5.4) in Eq. (2.9) we find that

$$\bar{f}_3 \to A \sec k_1 y \tag{5.5}$$

In order that the boundary condition (5.1) be satisfied we need, for $k_1 = \infty$,

$$A = 0. \tag{5.6}$$

Thus Eq. (5.4)

$$A = B = 0. \tag{5.7}$$

Subject to Eq. (5.7), we now rewrite the relevant equations for the top plate; they are

$$\begin{aligned} \bar{f}_1 &= -\frac{D k \cos ky}{i\beta\cos\alpha} + \frac{C k \sin ky}{i\beta\cos\alpha} \\ \bar{f}_3 &= 0 \end{aligned} \tag{5.8}$$

$$-i\omega\mu\bar{g}_1 = -i\beta\sin\alpha\,(C\cos ky + D\sin ky)$$

$$-i\omega\mu g_3 = -\frac{\beta^2\cos^2\alpha + k^2}{i\beta\cos\alpha}\,(C\cos ky + D\sin ky)$$

The use of the boundary conditions (5.1),(5.2),(5.3) of the Eqs. (5.8) and the pertinent Eqs. (3.10)-(3.15) lead to the following homogeneous system of equations

$$\begin{aligned} 0 &= C\cos k(d+t) + D\sin k(d+t) \\ A_3\cos\alpha\cos kd - B_3\sin\alpha\sin kd &= C k\sin kd - Dk\cos kd \\ A_3\frac{\omega^2\epsilon\mu}{k}\cos\alpha\sin kd + B_3 k\sin\alpha\cos kd &= -(\beta^2\cos^2\alpha + k^2)(C\cos kd + D\sin kd) \\ A_3\sin\alpha\cos kd + B_3\cos\alpha\sin kd &= 0. \end{aligned} \tag{5.9}$$

The constants C and D of Eqs. (5.8) have been suitably modified in order to suppress the term $i\beta\cos\alpha$. Eqs. (5.8) refer to the top plate.

For the bottom plate, it can be shown that the constants A_1 and B_1 of Eqs. (2.24)-(2.28) are both zero. If now in Eqs. (5.9) α, d and d + t

be replaced by their negatives and if the constants C_1 and D_1 of Eqs.(2.24)-(2.28) be taken so that

$$C_1 = -C\,, \qquad D_1 = D\,, \tag{5.10}$$

Eq. (5.9) remains unchanged.

Proceeding along the lines of Section 4, we see that in order that the system (5.9) have a non-trivial solution it is necessary and sufficient that the determinant formed by the coefficients of A_3, B_3, C and D be zero. That is

$$\begin{vmatrix} 0 & 0 & \cos k(d+t) & \sin k(d+t) \\ \cos\alpha\cos kd & -\sin\alpha\sin kd & k\sin kd & -k\cos kd \\ \frac{\omega^2\epsilon\mu}{k}\cos\alpha\sin kd & k\sin\alpha\cos kd & -(\beta^2\cos^2\alpha+k^2)\cos kd & -(\beta^2\cos^2\alpha+k^2)\sin kd \\ \sin\alpha\cos kd & \cos\alpha\sin kd & 0 & 0 \end{vmatrix} = 0 \tag{5.11}$$

The Eq. (5.11) may be reduced to

$$(k^2\tan^2\alpha - \omega^2\epsilon\mu\tan^2 kd) = (\omega^2\epsilon\mu + k^2\tan^2\alpha)\tan kd\tan kt \tag{5.12}$$

Letting $k = iK$ and $x = dK$, Eq. (5.12) reduces to

$$(\omega^2\epsilon\mu d^2\tanh^2 x - x^2\tan^2\alpha) = (\omega^2\epsilon\mu d^2 - x^2\tan^2\alpha)\tanh x\tanh Kt \tag{5.13}$$

If $t = 0$, Eq. (5.13) reduces to Brillouin's equation (3.18).

The use of Eq. (5.13) rather than Eq. (3.18) does not give any decided correction to the unknown x. For the range of frequency and helical angle of any practical importance, it turns out that the Eq. (3.18) has a root x such that tanh x is very nearly one. The right hand side of Eq. (5.13) is then calculated and found to be small independent of the choice of the thickness of the plate. Thus the use of successive approximations leads to no significant gain in using Eq. (5.13) over Eq. (3.18).

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